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Statistics of "Worms" in Isotropic Turbulence Treated on the Multifractal Basis

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As is well known, there appear innumerable, randomly posed, unsteady, vorticity-concentrated slender regions, which may be called "worms," moving around in isotropic turbulence with a high Reynolds number. The universal feature of instantaneous spatial distribution of these worms is formulated based on a multifractal structure of the energy dissipation rate previously investigated and on the assumption that a worm can be locally replaced by a Burgers vortex.

KEY WORDS: Worm; Burgers vortex; multifractal; intermittency; isotropic turbulence; inertial range of scale.

1. INTRODUCTION

Turbulence is a nonequilibrium random phenomenon in a classical fluid governed by typically nonlinear dynamics with dissipation. It is considered as a chaos with tremendous degrees of freedom. But at present, we have no statistical-mechanical tool to elucidate its structure. Many typical facts obtained by observations constitute the greater part of our knowledge of turbulence. For this reason, we present a statistical but phenomenological treatment of the turbulent vortical structure of a fluid, by utilizing a model of scale-similar multifractal measure of dissipation.

As a result of the recent development of direct numerical simulation (DNS) of decaying and forced isotropic turbulences using a supercomputer, it is well known[1–3] that there appear innumerable vorticity-concentrated slender regions in the fully-developed turbulent flow space. These regions are worm-like and have the diameter of the order of Kolmogorov

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scale η $(=(\nu^3/\varepsilon)^{1/4}$; ν: kinematic viscosity of fluid; ε: globally averaged energy dissipation rate per unit mass). We call them "worms."[1,3] Their length seems to be of the order of integral scale, i.e., the correlation scale of longitudinal velocity fluctuation from the viewpoint of DNS. But Novikov[4] pointed out, based on the Navier–Stokes dynamics, that a "vortex string" to be in turbulence scales with $LR_\lambda^{-3/5}$, where L is macroscale and R_λ is the Taylor-scale Reynolds number. A vortex string may be understood as his terminology for a worm, since it usually has a concentrated vorticity field vertical to its section. It is to be noted that this scale is larger than Taylor scale λ $(\cong 15LR_\lambda^{-1})$. On the other hand, Jimenez and Wray[5] showed that the axial vorticity correlation length in a worm scales with Taylor scale.

The multifractal structure[6] of energy dissipation rate in isotropic turbulence was introduced to explain the universal intermittency of turbulence. Many models for this have been proposed and compared with experiments and DNS since Kolmogorov.[7] Obviously, the spatial distribution of energy dissipation must be dynamically interrelated with the distribution of localized vorticity.[8] Therefore, it is natural to think that the multifractal structure of energy dissipation in the turbulent flow space is closely linked to the distribution or arrangement of worms with various radii and circulations around them. At the present stage, it is hard to derive the latter by analytical treatment of the first principle: the Navier–Stokes equation, but it may be possible to predict it based on the universal multifractal structure of energy dissipation, at least, in a qualitative way if a proper model for it is adopted. This is the aim of this paper.

Here we use the three-dimensional (3D) binomial Cantor set model, which was contrived by the author[9] in 1991 and represents quite well the intermittent or multifractal structure of energy dissipation obtained by a DNS with $R_\lambda \cong 100$ and 160.[10] Also this model, when combined with a generalization of the refined similarity hypothesis by Kolmogorov,[11] succeeded in predicting the probability density functions (PDF) of longitudinal velocity difference across any distance down to Kolmogorov scale in excellent comparison with experiments. We will utilyze here the fact that the multifractal of this model can give the spatial distribution of energy dissipation locally averaged over a domain of any scale down to Kolmogorov scale. In order to predict the statistics of worms, however, we need three additional bold postulates to simplify our treatment, which are described in the next section. Then we will estimate the PDF of worm radius, circulation and length, and the total characters of worms in a macroscale cube. Our estimation is still crude, but it contributes to our systematic understanding of the statistical structure of worms in turbulence, since there is no sound theory established on this matter yet.

2. WORKING POSTULATES

Postulate 1. The multifractal structure of energy dissipation is represented by the 3D binomial Cantor set model.

To view turbulence as a fractal structure was initiated by Frisch, Sulem, and Nelkin.[12] One essential improvement of the idea—to extend the fractal to a multifractal concept—is the 3D binomial Cantor set model by the present author,[9] who demonstrated that there is a reasonable agreement with the DNS result of decaying isotropic turbulence with $R_\lambda \simeq 100$ and 160.[10] It is to be noted that the multifractal is universal in the inertial range of scale irrespective of R_λ, whereas R_λ governs only the value of Kolmogorov scale under which the multifractal does not prevail.

In this model, energy dissipation serves as a multifractal measure in 3D space from macroscale L down to Kolmogorov scale η. If $L > \ell > r > \eta$, and we write energy dissipation locally averaged over a domain of scale r as ε_r, (therefore $\varepsilon = \varepsilon_L$), the stochastic variable $y = \varepsilon_r/\varepsilon_\ell$ distributes itself spatially with the PDF:

$$p(y; r/\ell) = 2^{-\Omega} \sum_{k=0}^{\Omega} {}_\Omega C_k \, \delta(y - B^{\Omega-k} C^k) \tag{1}$$

where $\Omega = -\ln(r/\ell)/\ln A$, $A = 2^{1/d}$ (d: spatial dimension; we take $d = 3$), and B, $C = 1 \pm (2^{\mu/d} - 1)^{1/2}$ (that is, 1.2175 and $C = 0.7825$ when we take the second-order intermittency exponent $\mu \equiv \mu(2) = 0.20$, the currently accepted value).

In particular, when $r/\ell = 1/2$ (half splitting), we have

$$p(y; 1/2) = (1/8)[\delta(y - B^3) + 3\delta(y - B^2 C) + 3\delta(y - BC^2) + \delta(y - C^3)] \tag{2}$$

Postulate 2. The sectional vorticity field of a worm can be approximately considered as that of a Burgers vortex with the same circulation.

A worm in reality is never exactly axisymmetric, but the azimuthal average of the vorticity field in a section of any worm is very close to that of a Burgers vortex with the same circulation and a comparable longitudinal strain.[5, 13]

The vorticity field of a Burgers vortex as an exact solution of the Navier–Stokes equation is given as

$$\omega(r) = \frac{\alpha \Gamma}{4\pi v} \exp\left(-\frac{\alpha r^2}{4v}\right) \tag{3}$$

where r is the radius around the central axis, α the longitudinal strain, and Γ the circulation around the vortex for $r \to \infty$. We define the Burgers radius as $r_B = 2(\nu/\alpha)^{1/2}$, and assume that r_B is m times the local Kolmogorov length $\eta' = (\nu^3/\varepsilon_{r'})^{1/4}$; r' is the smallest scale over which we can consider the local character of turbulence. Naturally, this scale r' should involve the whole inertial range of scale.

As a result, we have

$$r_B/\eta = m(\varepsilon_{r'}/\varepsilon_L)^{-1/4} \tag{4}$$

The proper values of m and r' will be fixed empirically later.

Postulate 3. The spatial distribution of enstrophy locally averaged over a domain of scale much larger than Kolmogorov scale can be approximated by that of the energy dissipation.

This means that approximately

$$\varepsilon_r/\varepsilon_L = (\omega^2)_r/(\omega^2)_L \tag{5}$$

where $(\omega^2)_r/2$ is enstrophy averaged over a domain of scale r. We note that Nelkin[13] recently gave a complete theoretical support, based on the Navier–Stokes equation, for the equivalence between the multifractal structures of energy dissipation and enstrophy in the high Reynolds number limit.

In fact, we observed in DNS of a decaying isotropic turbulence with $R_\lambda \cong 160$,[14] that the correlation coefficient of ε_r and $(\omega^2)_r$ ranges from 0.99 to 0.96 in the inertial range of r and drops, at most, to 0.85, 0.65 and 0.48 for r equal to about 4, 2 and 1 times Kolmogorov scale. This supports the Postulate very well even for a finite R_λ, meaning that both quantities are localized rather near each other in space so long as we view them in rough resolution of more than 8 times Kolmogorov scale, whether the values of both sides of Eq. (5) are larger than 1 or not, and then that enstrophy makes a measure with almost the same structure as dissipation at least in the inertial range of r.

From Eq. (3) we can obtain

$$(\omega^2)_{2r_B} = \Gamma^2(1-e^{-2})/(\pi^2 r_B^4) \tag{6}$$

Equations (4), (5) and (6) lead to

$$\frac{\Gamma}{\nu} = \frac{\pi m^2}{(1-e^{-2})^{1/2}} \left(\frac{\varepsilon_{2r} \propto}{\varepsilon_{r'}} \right)^{1/2} \tag{7}$$

which may be called the Burgers Reynolds number R_Γ.

Postulate 4. The spatial distributions of enstrophy and dissipation locally averaged over a domain of scale near the local Burgers radius must be locally almost two-dimensional.

This is required by the morphologic nature of the the Burgers vortex but puts a strong restriction to application of Postulate 3 to a domain of small scale, at least, less than Taylor scale. Accordingly, for those scales the splitting of a cube into 8 subcubes in the measure breakdown process of Postulate 1 should be interpreted as the splitting of a cylinder into 8 parallel cylinders in it. Thus, we take a simple idealization that a Taylor-scale wide cube splits into $\lambda^2/(2r_B)^2$ cylinders each $2r_B$ wide and λ long, after n breakdown processes repeated; $n=\log_8[\lambda^2/(2r_B)^2]$. Only a cylinder with the maximum enstrophy among a coagulation of 8 cylinders which has occurred in the final breakdown process is considered as the core of a worm. So 1/8 of the $\lambda^2/(2r_B)^2$ cylinders can be worms. This thought may be an extreme modeling. In fact, such an orderly alignment of straight cylinders in a Taylor-scale wide cube is hardly ever observed, but what we can expect to see at most is an entanglement of more or less deformed cylinders, keeping the same topology as the straight cylinders in the cube. So, our hope is that this topology embodies the essence of the phenomenon even though qualitatively.

Since the maximum dissipation should occur near the cylinder with the maximum enstrophy in the coagulation as is expected in comparison with a Burgers vortex, we have with Eqs. (2) and (5) in mind

$$\nu\langle\omega^2\rangle_{2r_B}=\varepsilon_{2r_B}=B^3(\varepsilon_{\lambda(1/2)^{n-1}}/\varepsilon_\lambda)(\varepsilon_\lambda/\varepsilon_{r'})(\varepsilon_{r'}/\varepsilon_L)\,\varepsilon_L \tag{8}$$

for a worm. We will introduce local Taylor scale in place of λ in the later sections, like local Kolmogorov scale.

3. PDF OF WORM RADIUS

Now let us calculate the number of worms with the radius r_B, $N(r_B)$. Postulate 4 suggests that

$$N(r_B)=(1/8)(\lambda'^2/(2r_B)^2)(L^3/\lambda'^3) \tag{9}$$

where λ' is the local Taylor scale, which is related to $\varepsilon_{r'}$ by $\lambda'^2\varepsilon_{r'}=\lambda^2\varepsilon_L$ $(=15\nu\langle u^2\rangle)$.[15] ($\langle u^2\rangle$: square-average of longitudinal velocity fluctuation.)

After taking into account(15) $\eta/L \cong 15^{3/4} R_\lambda^{-3/2}$ and $\lambda/L \cong 15 R_\lambda^{-1}$, and putting $y = \varepsilon_{r'}/\varepsilon_L$, Eq. (9) can be rewritten as

$$N(r_B) = y R_\lambda^4 / (32 \times 15^{5/2} m^2) \tag{10}$$

where we note from Eq. (4)

$$y = m^4 (r_B/\eta)^{-4} \tag{11}$$

This shows that both $N(r_B)$ and the total number of worms are in proportion to R_λ^4, since the scale r' is not considered to be dependent on R_λ.

Thus, the PDF of worm to have radius r_B is given as

$$P(r_B/\eta) = 4m^4 (r_B/\eta)^{-5}\, yp(y; r'/L) \Big/ \int yp(y; r'/L)\, dy \tag{12}$$

together with Eq. (11). It is easy to prove that the denominator is exactly unity for any r'/L, based on Eq. (1). (This means that our multifractal of dissipation is measure-conserving.(6)) In a Cantor set model, as given as Eq. (1), generally we have a discrete PDF as $p(y; r/\ell)$. However, as $P(r_B/\eta)$ in reality is expected to be continuous, we had better make a continuous version of the $p(y; r/\ell)$. Special care is necessary to do so, because the intervals of discrete values of y vary with k; the kth value of p must be changed so that the integrated area $p\, dy$ over half the interval between the $(k-1)$th and $(k+1)$th y may be equal to the original discrete kth value of p. With this care, we have representative points on the continuous versions of the $p(y; r/\ell)$ for $r/\ell = 1/8$ and $1/128$ as shown in Fig. 1(a) and (b), respectively. The solid lines indicate the correspondiong curves obtained from the lognormal model of Kolmogorov for $\mu = 0.2$:

$$p(y; r/\ell) = \exp\{-[\ln y - m(r/\ell)]^2 / [2s(r/\ell)^2]\} / [(2\pi)^{1/2}\, ys(r/\ell)]$$

with

$$s(r/\ell) = [\mu \ln(\ell/r)]^{1/2}, \qquad m(r/\ell) = -s(r/\ell)^2/2 \tag{13}$$

Hence one can recognize that both models are surprisingly close to each other, so long as direct comparison of the main parts of the PDF's is concerned; although the tails of the lognormal PDF extend to $y = 0$ and infinity while our PDF does not. (This small difference affects high-order moments which is not considered here). Therefore we will use Eq. (13) as a surrogate only when the continuous version of Eq. (1) is desirable. Of course, the denominator of Eq. (12) is unity in this case, too.

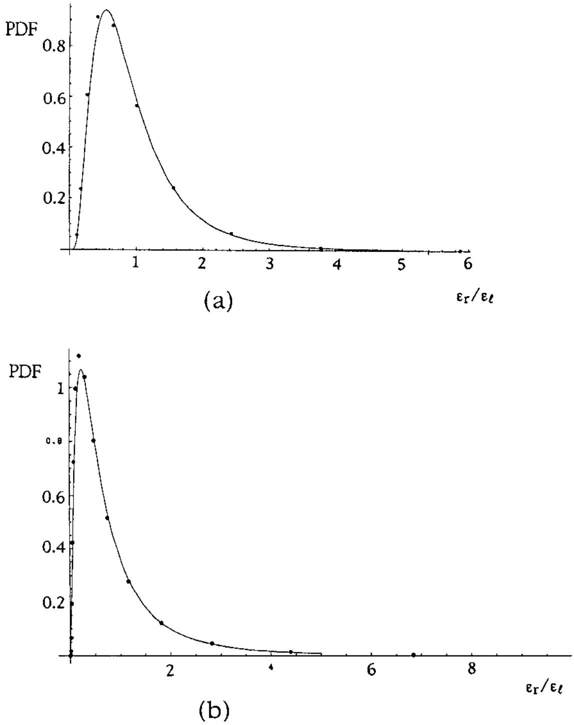

Fig. 1. The PDFs of dissipation ratio ($\varepsilon_r/\varepsilon_\ell$) based on the continuous version of the 3D binomial Cantor set model (dots) and the lognormal model (solid) for (a) scale ratio $(r/\ell) = 1/8$ and (b) $= 1/128$.

In Fig. 2 we show the $P(r_B/\eta)$ thus obtained for $r'/L = 1/8$ and $m = 4$, in comparison with Jimenez *et al.*'s DNS results.[3, 5] Our $P(r_B/\eta)$ is insensitive to r'/L so that it is almost unchanged even for $r'/L = 1/16$, but it depends appreciably on m; its peak moves to the right with increasing m so that the peak comes at $r_B/\eta \cong 4$ if $m = 4.2$. It is common to Jimenez *et al.*'s results that $P(r_B/\eta)$ is independent of R_λ, while there is a considerable difference in the form of $P(r_B/\eta)$ between their result and ours even though the location of the peak is similar. Probably, the difference is mostly caused by the fact that they sampled only worms with strong vorticity (much stronger than the global average of vorticity) in their DNS; while there must be many worms with weak vorticity such as seen in our DNS result,[13] which they did not count.

Our choice of $r'/L = 1/8$ and $m = 4$ will be retained from now on. Since $r'/L = 1/8$ is just near the upper rim of the inertial range,[11] this value is

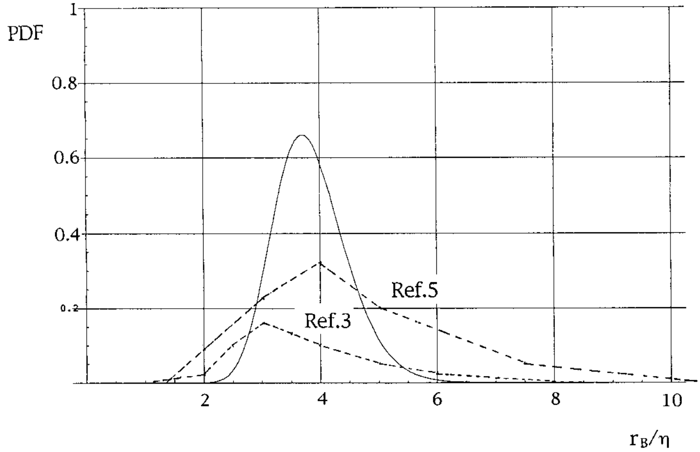

Fig. 2. The PDFs of worm radius (r_B/η) by the present theory (solid line) compared with Jimenez *et al.*'s DNS results[3, 5] (dotted lines; sketches of what were obtained for the highest R_λ in each reference).

reasonable for Postulate 2. $m=4$ is the best as a one-digit value, judging from Fig. 2 and considering that our aim is a crude estimation.

4. PDF OF WORM CIRCULATION

From Eqs. (7) and (8) we can write

$$R_\Gamma = aB^{3/2}y'^{1/2}, \qquad \text{or} \quad y' = R_\Gamma^2/(a^2B^3) \tag{14}$$

where $a=\pi m^2/(1-e^{-2})^{1/2}$ and $y'=\varepsilon_{\lambda'(1/2)^{n-1}}/\varepsilon_{r'}$; we take into account local Taylor scale $\lambda'=\lambda(\varepsilon_L/\varepsilon_{r'})^{1/2}$. According to Postulate 4, n is given as

$$n=\log_8[\lambda^2 y^{-1}/(2m\eta y^{-1/4})^2]=\log_8[15^{1/2}R_\lambda y^{-1/2}/(4m^2)] \tag{15}$$

with Eq. (11) and $\lambda/\eta=15^{1/4}R_\lambda^{1/2}$ (ref. 15) in mind.

We know that y and y' should distribute according to the PDF of Eq. (1) with scale ratio 1/8 and

$$\lambda'(1/2)^{n-1}/r' = 4^{7/3}\times 15^{5/6}m^{2/3}R_\lambda^{-4/3}y^{-1/3} \tag{16}$$

respectively. Therefore, we can construct the PDF of R_Γ as follows;

$$P(R_\Gamma)=2R_\Gamma/(a^2B^3)\int p(R_\Gamma^2/(a^2B^3);\, 4^{7/3}\times 15^{5/6}m^{2/3}R_\lambda^{-4/3}y^{-1/3}) \times p(y;\, r'/L)\, dy \tag{17}$$

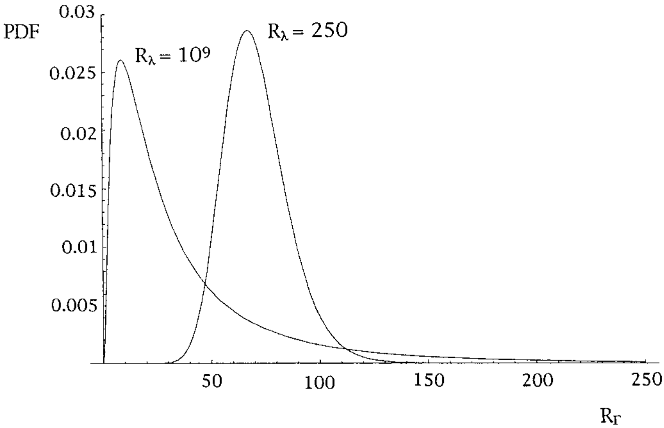

Fig. 3. The PDFs of worm circulation (R_Γ) for $R_\lambda = 250$ and 10^9.

With $m = 4$ and $r'/L = 1/8$, we obtain $P(R_\Gamma)$ for $R_\lambda = 250$ and 10^9 as in Fig. 3. For convenience of calculation, we have used the lognormal approach to $p(y'; \lambda'(1/2)^{n-1}/r')$. Hence we can see that R_Γ distributes much less widely in the case of $R_\lambda = 250$ than 10^9. But the peak of the PDF occurs at a much larger value of R_Γ in the former case than the latter, implying that fine-scale worms of small circulations overwhelmingly increase as R_λ increases, even though the population of worms with very large circulation increases with R_λ, as well. It is evident from Fig. 3 that the PDF's in our result never collapse for different R_λ's, even if the horizontal axis is normalized by $R_\lambda^{1/2}$; in contrast to Jimenez *et al.*'s DNS result.[3] Whether $P(R_\Gamma/R_\lambda^{1/2})$ should collapse or not is discussed in next section.

For R_λ less than 213.47, we have the possibility that $\lambda'(1/2)^{n-1}/r' > 1$, i.e., there cannot be any worm in a subcube of side r' subject to this condition, as is easily calculated from Eqs. (1) (with $\Omega = 9$) and (16). Such a subcube should not be counted in Eq. (17). The normalized $P(R_\Gamma)$ for such a case is similar to that for $R_\lambda = 250$ in Fig. 3 but more Gaussian and less wide, the peak being deviated very slightly more rightward. And we have no worms anywhere for R_λ less than 78.95 in the present theory, while we often find worms by DNS for such a low R_λ. This gap seems to be due to the idealization contained in our four Postulates which presuppose the existence of the inertial range of $r < r'$.

5. CAN $R_\Gamma/R_\lambda^{1/2}$ BE A SIMILARITY VARIABLE FOR PDF?

Let us notice the relation:

$$R_\Gamma/R_\lambda^{1/2} \cong \Gamma/[15^{1/4}(L\varepsilon_L^{1/4})^{1/3}\,\nu^{3/4}] \tag{18}$$

by definition. In order for this number to be a finite variable for any R_λ or ν, we must have Γ such that

$$\Gamma \sim (L\varepsilon_L^{1/4})^{1/3}\,\nu^{3/4} \qquad \text{for} \quad \nu \to 0 \tag{19}$$

On the other hand, we have from Eqs. (4), (6) and (19)

$$(\omega^2)_{2r_B} = \Gamma^2(1-e^{-2})/(\pi^2 r_B^4) \sim L^{2/3}\varepsilon_L^{7/6} y/\nu^{3/2} \sim R_\lambda \varepsilon_L/\nu \tag{20}$$

while

$$(\omega^2)_L = \varepsilon_L/\nu \tag{21}$$

This shows that, as $\nu \to 0$, the average enstrophy in the core of a worm would be R_λ times larger than the global average of enstrophy, since y is independent of R_λ or ν (and finite according to $p(y; 1/8)$ given by Eq. (1)). This is consistent with Jimenez *et al.*'s consideration.[3, 5]

Let us consider the average circulation of a worm on the basis of our theory. From Eq. (17) we have

$$\int R_\Gamma P(R_\Gamma)\, dR_\Gamma$$

$$= (a^2B^3)^{1/2} \iint y'^{1/2} p(y'; 4^{7/3} \times 15^{5/6} m^{2/3} R_\lambda^{-4/3} y^{-1/3})\, dy'\, p(y; r'/L)\, dy$$

$$= (a^2B^3)^{1/2} \int (4^{7/3} \times 15^{5/6} m^{2/3} R_\lambda^{-4/3} y^{-1/3})^{-\ln[(\sqrt{B}+\sqrt{C})/2]/\ln A}\, p(y; r'/L)\, dy$$

$$\sim B^{3/2} R_\lambda^{-0.0347} \int y^{-0.00869} p(y; r'/L)\, dy \tag{22}$$

since $-\ln[(\sqrt{B}+\sqrt{C})/2]/\ln A = 0.02606$, so that on the average

$$\Gamma \sim \nu^{1.017} \tag{23}$$

where we used the relation: $R_\lambda \sim \nu^{-1/2}$ suggested by the order-of-magnitude equality (20). This gives a different scaling from Eq. (19). This is the reason why our result does not have a similar behavior with respect to the variable $R_\Gamma/R_\lambda^{1/2}$. Therefore the average enstrophy must be much smaller than the average Jimenez *et al.* obtained, as is inferred from Eq. (20). This indicates that there are many worms with weak vorticity, even weaker than the global average, which Jimenez *et al.* totally neglected. We actually observed many such worms in relatively low-vorticity and consequently low-dissipation regions with $\varepsilon_{r'} < \varepsilon_L$ in a DNS decaying isotropic turbulence.[13] This is a necessary result stemming from the global nature of the multifractal measure.[9] It seems to the author to be more general to consider all kinds of worms linked to the universal multifractal measure of dissipation.

Let us investigate the maximum of enstrophy of a worm in our theory. Obviously from Eq. (8), we have the maximum:

$$\begin{aligned}\mathrm{Max}[(\omega^2)_{2r_B}] &= (B^3\varepsilon_L/\nu)\, B^{\Omega}\\ &= (B^3\varepsilon_L/\nu)[15^{1/6}y^{-1/6}R_\lambda^{4/3}/(120m^{2/3})]^{\ln B/\ln A}\end{aligned} \tag{24}$$

where $\Omega = -\ln(\varepsilon_{\lambda'(1/2)^{n-1}}/\varepsilon_L)/\ln \mathrm{A}$ and which is still stochastic in y. Since generally we have the intermittency relation of a multifractal[6]:

$$\langle(\varepsilon_r/\varepsilon_\ell)^q\rangle = \int y^q p(y; r/\ell)\, dy = (r/\ell)^{-\mu(q)} \tag{25}$$

(where angular bracket means ensemble average and $\mu(q)$ are called intermittency exponents), the factor of $y^{-(\ln B/\ln A)/6}$ may be replaced by its average in this case: $(1/8)^{-\mu(-(\ln B/\ln A)/6)}$. Taking into account that $\ln B/\ln A = 0.852$, we can see that the maximum scales with $\nu^{-1.568}$. This is slightly larger than what is expected from Eq. (20), that is $\nu^{-1.5}$. The minimum is obtained by replacing the exponent $\ln B/\ln A$ by $\ln C/\ln A$, so that it scales with $\nu^{-0.292}$. Our theory then suggests that $(\omega^2)_{2r_B}$ of worms does not scale uniquely with $\nu^{-1.5}$ [like (20)] but the scaling exponent ranges over a wide region including -1.5. The worms with strong vorticiy that Jimenez *et al.*'s sampled, say "hard worms," seem to be the group of worms whose scaling exponents of enstrophy are close to -1.5.

6. NUMBER AND LENGTH OF WORMS

It is obvious from Eq. (10) that the expectation value of the total number of worms is

$$N_T = R_\lambda^4/(32 \times 15^{5/2} m^2) \tag{26}$$

since $\int yp(y; r'/L)\, dy = 1$. For $R_\lambda = 100$ we have $N_T \cong 224$, a value which is comparable with that obtained from a current DNS,[5] although we cannot expect an accurate prediction because our theory is based on simple postulates. As is known from $\int y^2 p(y; r'/L)\, dy = [(B^2 + C^2)/2]^9 = 1.516$ (using Eq. (1) with $r'/L = 1/8$), the standard deviation is $0.718 N_T$. It is also pointed out that the present theory may not necessarily fit the turbulence with such a low R_λ in which case we can hardly see a sufficient inertial range.

According to Postulate 4, a worm must be local-Taylor-scale long at least. The $N(r_B)$ worms counted in Eq. (10) should have the length of λ', and then the PDF of worm length can be written as

$$P(\lambda'/L) = 30R_\lambda^{-2}(\lambda'/L)^{-3}\, yp(y; r'/L) \tag{27}$$

with $\lambda' = \lambda y^{-1/2} = 15LR_\lambda^{-1}y^{-1/2}$ in mind. Thus, the average of λ'/L is given as

$$\int (\lambda'/L)\, P(\lambda'/L)\, d(\lambda'/L) = 15R_\lambda^{-1} \int y^{1/2}p(y; r'/L)\, dy \tag{28}$$

meaning that the average of worm length scales with LR_λ^{-1} or $\eta R_\lambda^{1/2}$. This is consistent with the observation of Jimenez *et al.*[5] The average is evaluated as $15LR_\lambda^{-1}[(B^{1/2}+C^{1/2})/2]^9 = 14.21LR_\lambda^{-1}$.

The expected total length of worms is the integration of $\lambda' N(r_B)$ with the measure of y, which is

$$\int 15LR_\lambda^{-1}y^{-1/2}yR_\lambda^4/(32\times 15^{5/2}m^2)\; p(y; r'/L)\, dy$$

$$= LR_\lambda^3 \int y^{1/2}p(y; r'/L)\, dy/(32\times 15^{3/2}m^2) = 0.000510LR_\lambda^3/m^2 \tag{29}$$

using Eq. (10). With $m = 4$, we have $31.9L$ for $R_\lambda = 100$. This increases in proportion to R_λ^3.

The volume of a worm may be estimated as $(2r_B)^2\,\lambda'$, so that the expected total volume of worms is given as the integration of $4r_B^2\lambda' N(r_B)$, which is

$$4\int L^2(15^{3/4}mR_\lambda^{-3/2}y^{-1/4})^2 \times 15LR_\lambda^{-1}y^{-1/2}yR_\lambda^4/(32\times 15^{5/2}m^2)$$

$$\times\, p(y; r'/L)\, dy = L^3/8 \tag{30}$$

with the help of Eq. (4). This shows that the volume ratio of all worms is invariant to R_λ as 1/8 if roughly estimated based on our simple theory. Therefore the diameter of a worm scales with $LR_\lambda^{-3/2}$. It is clear that 1/8 comes from the factor 1/8 in Postulate 4. In this thinking, the probability of worm-core appearance might be underestimated, since a cylinder without the maximum enstrophy in a coagulation may sometimes happen to be a core of worm.

The tendencies of the above two equations are different from what Jimenez *et al.*[5] concluded. A main reason is that their treatment focused on hard worms, and another may be that the DNS study is limited only to turbulence of small Reynolds numbers; the grand trend in fine structure of turbulence of high $R_\lambda s$ may be deviated from an extension from the trend observed for small $R_\lambda s$. Our theory suggests that the background and weak-vorticity regions in turbulence assigned by Jimenez *et al.*[3] includes still many large worms, say "soft worms."[13] Recently, Kida *et al.*[16] estimated the volume ratio to be 42% for $R_\lambda = 46$. This value, however, looks too large to extend towards higher $R_\lambda s$.

7. CONCLUSION

We have tried a new statistical description to explain the instantaneous spatial distribution of all kinds of (hard as well as soft) worms in isotropic turbulence based on the universal multifractal measure of dissipation rate previously investigated (Postulate 1). Three other Postulates (in Section 2) play important roles for this achievement. Postulates 2 and 3 are reasonable since we have suitable dynamical backgrounds and the DNS results to support them. Postulate 4 is the boldest but necessary for approaching the idealized worm structure with the extensive use of Postulate 3. Our trial is a systematic expression of the conception that the total arrangement of worms should be interrelated with intermittency or multifractality in turbulence. This is novel in implying the structural importance of all kinds of worms arranged with a special harmony, in contrast to a popular thinking that the weak vorticity field outside hard worms is simply random and incoherent. Even if we might thereby offer nothing beyond a rough overview, this treatment would give some insight into the vortical structure of turbulence closely linked to the geometry of self-similar dissipation measure. At present, however, there are no high R_λ experimental data of worms available to compare with our prediction, beyond the DNS of Jimenez *et al.* which was discussed here. We expect our work will give some impetus to further research on the statistics of worms for high enough R_λ.

There are various models for the multifractal dissipation measure in isotropic turbulence which may substitute for the 3D binomial Cantor set model in Postulate 1. It seems to the author that any other model which can give about the same intermittency exponents $\mu(q)$ in low orders (up to at most $q \cong 4$) would work to yield almost the same qualitative result as we have now, because the principal part of the PDF of dissipation is determined by its low-order moments. The lognormal model is just such. But from the viewpoint of simplicity and convenience in calculation, the present model is one of the best. No model has ever been derived from the Navier–Stokes equation, however. This remains a task to be finally done.

We have used local Taylor scale here to estimate the length of a worm. There is, however, another scale available for this purpose proposed by Novikov. We briefly describe in the Appendix what else can be predicted if Novikov's scale is used.

APPENDIX. CASE OF NOVIKOV'S SCALE

Novikov[4] proposed a new scale as the length of worms. That is

$$l_s = 1.66 L Re^{-3/10} = 1.66 \times 15^{3/10} L R_\lambda^{-3/5} \tag{A1}$$

Let us introduce the local Novikov scale l'_s with the local $R'_\lambda = R_\lambda(\lambda'/\lambda) = R_y\, y^{-1/2}$, and then it can be written as

$$l'_s = 1.66 \times 15^{3/10} L R_\lambda^{-3/5} y^{3/10} \tag{A2}$$

We summarize what change happens in the result by replacing λ' by l'_s.

As a result we have more complicated forms than Eq. (10) and (26),

$$N(r_B) = y^{1/5} R_\lambda^{18/5}/(32 \times 1.66 \times 15^{9/5} m^2) \tag{A3}$$

$$N_T = [(B^{1/5} + C^{1/5})/2]^9\, R_\lambda^{18/5}/(32 \times 1.66 \times 15^{9/5} m^2) \tag{A4}$$

The number of worms decreases considerably as compared with that in the text. For example, for $R_\lambda = 100$ we have $N_T \cong 138$.

The PDF of worm radius becomes

$$P(r_B/\eta) = 4m^4 (r_B/\eta)^{-5}\, y^{1/5} p(y; r'/L) \Big/ \int y^{1/5} p(y; r'/L)\, dy \tag{A5}$$

In place of Eq. (15) we have

$$n = \log_8[1.66^2 \times 15^{-31/15} R_\lambda^{9/5} y^{-11/10}/(4m^2)] \tag{A6}$$

and hence the scale ratio (16) is changed to

$$l'_s(1/2)^{n-1}/r' = 4^{7/3} \times 1.66^{1/3} \times 15^{89/90} m^{2/3} R_\lambda^{-6/5} y^{-1/15} \tag{A7}$$

Thus, the PDF of R_Γ should be constructed by inserting (A7) as the scale ratio for y' into Eq. (17).

Using the relation of l'_s/L and y in (A2), the PDF of worm length is given in a similar way to (A5) as

$$P(l'_s/L) = 2/(9 \times 1.66^{10/3})(l'_s/L)^{7/3}\, R_\lambda^2 y^{1/5} p(y; r'/L) \Big/ \int y^{1/5} p(y; r'/L)\, dy \quad \text{(A8)}$$

The expected total length of worms is the integration of $l'_s N(r_B)$ with the measure of y, which is

$$\int 1.66 \times 15^{3/10} L R_\lambda^{-3/5} y^{3/10} y^{1/5} R_\lambda^{18/5} / (32 \times 1.66 \times 15^{9/5} m^2)\; p(y; r'/L)\, dy$$

$$= L R_\lambda^3 \int y^{1/2} p(y; r'/L)\, dy / (32 \times 15^{3/2} m^2) \quad \text{(A9)}$$

This is in complete accord with Eq. (29).

The expected total volume of worms is given as the integration of $4 r_B^2 l'_s N(r_B)$, which is, using the integrand of (A9),

$$4 \int L^2 (15^{3/4} m R_\lambda^{-3/2} y^{-1/4})^2\, L R_\lambda^3 y^{1/2} p(y; r'/L)\, dy / (32 \times 15^{3/2} m^2) = L^3/8 \quad \text{(A10)}$$

This also agrees Eq. (30).

Therefore, the replacement of λ' by l'_s causes an appreciable difference from the argument in the text, except for (A9) and (A10). It may be interesting to note that if we take r' (assumed as $L/8$ in this paper) in place of λ', the difference to be caused will be more appreciable, particularly in the R_λ-dependence of N_T—which would be in proportion to R_λ^3. In order to judge which scale is real as the worm length, however, very reliable experimental or DNS data of the R_λ-dependence of N_T for high R_λs will be necessary.

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